Pergamon

J. Appl. Maths Mechs, Vol. 59, No. 3, pp. 387–394, 1995

0021–8928/95 \$24.00+0.00

0021–8928(95)00046-1

A MODIFICATION OF THE PARALLEL-APPROACH METHOD†

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(*Received* 13 *May* 1993)

A modification of the parallel-approach method (or method of consecutive leads) [1, 2] is investigated based on solving an inverse dynamical problem: a motion given by the parallel-approach method is taken to be the perfect state of the mutual motion of an object and target, and the dynamics of a transition process is set up whereby the perfect state is re-established after being disrupted. The control forces are calculated from the equations of motion of the object and the prescribed equation for the transition process.

1. DESCRIPTION OF THE ALGORITHM. THE EQUATIONS OF MOTION

We introduce the following notation (Fig. 1): $\mathbf{R}$ and $\mathbf{R}_0$ are respectively the radius vectors of the target and object relative to the origin of an inertial system of coordinates, $\mathbf{r} = \mathbf{R} - \mathbf{R}_0$ is the radius vector of the target with respect to the object, $\mathbf{d}$ is the projection of the vector $\mathbf{r}$ onto the plane perpendicular to the vector $\mathbf{v} = \dot{\mathbf{r}}$, $\mathbf{e}$ is the unit vector collinear with the vector $\mathbf{d}$, and $d = |\mathbf{d}|$ is the distance between the target and the line along which the relative velocity vector of the object instantaneously lies. Here and below the length of the vector $\mathbf{x}$ is denoted by x.

The dynamics of the object are described by the equation

$$\ddot{\mathbf{R}}_0 = \mathbf{u} + \mathbf{g} \tag{1.1}$$

where $\mathbf{u}$ is the controlling acceleration vector and $\mathbf{g}$ is the acceleration due to external forces (such as gravity). The object is controlled according to the following rules

(A) the vector $\mathbf{u}$ is collinear with the vector $\mathbf{d}$;

(B) the quantity d satisfies the differential equation

$$\dot{d} + \alpha d = 0 \tag{1.2}$$

where $\alpha > 0$ is a constant characterizing the rate of the transition process.

The quantity d has thus been chosen as a measure of the deviation from the "perfect" parallel-approach motion in which the relative velocity vector $\mathbf{v} = \dot{\mathbf{R}}_0 - \dot{\mathbf{R}}$ is directed at the target. If initially $d = 0$, then from Eq. (1.2) $d(t) \equiv 0$ and the given homing algorithm is identical with the parallel-approach method. We shall assume that at the initial time t_0 the angle ϕ between the vectors $\mathbf{v}$ and $\mathbf{r}$ is acute, i.e. $0 < \phi < \pi/2$. We therefore have $(\mathbf{v}(t_0), \mathbf{r}(t_0)) > 0$ and $d(t_0) \neq 0$. From Eq. (1.2) it follows that $d(t) \neq 0$ when $t \geqslant t_0$. We shall also assume that the relative velocity $\mathbf{v}(t)$ satisfies the condition $\upsilon_1 \geqslant \upsilon(t) \geqslant \upsilon_0 > 0$, where υ_0, υ_1 are constants, and the vector functions $\mathbf{g}$ and $\ddot{\mathbf{R}}$ are bounded.

Using Eqs (1.1) and (1.2) and rules A and B to determine the control, we find the vector $\mathbf{u}$. We have (Fig. 1)

$$\mathbf{d} = \mathbf{r} - \upsilon^{-2}(\mathbf{r}, \mathbf{v})\mathbf{v}, \quad d = (r^2\upsilon^2 - (\mathbf{r}, \mathbf{v})^2)^{1/2}$$

where $(\mathbf{r}, \mathbf{v})$ is the scalar product of the vectors $\mathbf{r}$ and $\mathbf{v}$.

We differentiate d with respect to time

†*Prikl. Mat. Mekh.* Vol. 59, No. 3, pp. 410–418, 1995.

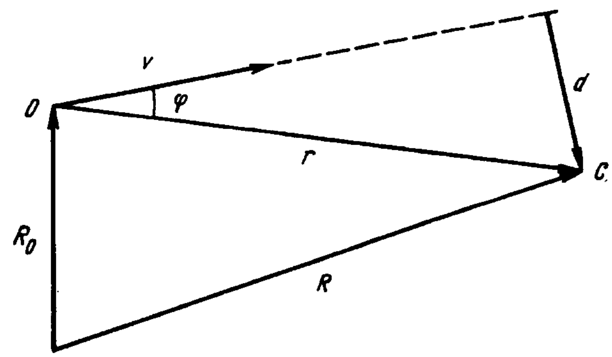

Fig. 1.

$$\dot{d} = \frac{(\mathbf{r},\mathbf{v})^2(\mathbf{v},\dot{\mathbf{v}}) - \upsilon^2(\mathbf{r},\mathbf{v})(\mathbf{r},\dot{\mathbf{v}})}{\upsilon^3(r^2\upsilon^2 - (\mathbf{r},\mathbf{v})^2)^{1/2}}$$

and note that the denominator does not vanish since $d \neq 0$. We substitute d and $\dot{d}$ into Eq. (1.2). After some reduction we obtain

$$(\mathbf{r},\mathbf{v})^2(\mathbf{v},\dot{\mathbf{v}}) - \upsilon^2(\mathbf{r},\mathbf{v})(\mathbf{r},\dot{\mathbf{v}}) + \alpha r^2\upsilon^4 - \alpha\upsilon^2(\mathbf{r},\mathbf{v})^2 = 0 \tag{1.3}$$

By assumption A the controlling acceleration vector $\mathbf{u}$ is directed along $\mathbf{d}$, i.e. $\mathbf{u} = u\mathbf{e}$. Using the notation introduced above we rewrite Eq. (1.1) in the form

$$\dot{\mathbf{v}} = u\mathbf{e} + \mathbf{g} - \ddot{\mathbf{R}} \tag{1.4}$$

The vector $\mathbf{e}$ is expressed in terms of $\mathbf{r}$ and $\mathbf{v}$ as follows:

$$\mathbf{e} = \frac{\upsilon^2\mathbf{r} - (\mathbf{r},\mathbf{v})\mathbf{v}}{\upsilon(r^2\upsilon^2 - (\mathbf{r},\mathbf{v})^2)^{1/2}} \tag{1.5}$$

Substituting expressions (1.4) and (1.5) into Eq. (1.3) and using $(\mathbf{v}, \mathbf{e}) = 0$, we obtain an equation from which we find the magnitude of the control

$$u = \frac{(\mathbf{r},\mathbf{v})^2(\mathbf{v},\mathbf{W}) - \upsilon^2(\mathbf{r},\mathbf{v})(\mathbf{r},\mathbf{W}) + \alpha r^2\upsilon^4 - \alpha v^2(\mathbf{r},\mathbf{v})^2}{\upsilon(\mathbf{r},\mathbf{v})(r^2\upsilon^2 - (\mathbf{r},\mathbf{v})^2)^{1/2}}$$

where $\mathbf{W} = \mathbf{g} - \ddot{\mathbf{R}}$. Equation (1.4) for the motion of the object relative to the target acquires the closed form

$$\ddot{\mathbf{r}} = \left[\frac{(\upsilon^2\mathbf{r} - (\mathbf{r},\mathbf{v})\mathbf{v},\mathbf{W})}{\upsilon^2(r^2\upsilon^2 - (\mathbf{r},\mathbf{v})^2)} - \frac{\alpha}{(\mathbf{r},\mathbf{v})}\right](\upsilon^2\mathbf{r} - (\mathbf{r},\mathbf{v})\mathbf{v}) - \mathbf{W} \tag{1.6}$$

Hence, in order to calculate the control using the given algorithm, it is necessary to know the radius vector of the target relative to the object, their relative velocity, and also the absolute target acceleration $\ddot{\mathbf{R}}$ and the acceleration due to the external forces $\mathbf{g}$.

2. INVESTIGATION OF THE ALGORITHM. TRAJECTORY BEHAVIOUR

To check the effectiveness of the algorithm it is necessary to verify that the object reaches a sufficiently small neighbourhood of the target. We shall show that with special assumptions and when there are no restrictions on the control, given enough time the object will arrive in an arbitrarily small neighbourhood of a point target. This also solves the problem of the impact of the object on a target of finite size.

We consider the following quantities: the distance $r(t)$ between the object and the target, the cosine $x(t) = (\mathbf{r}, \mathbf{v})/\upsilon(t) = \cos\varphi$ of the angle between the vectors $\mathbf{v}$ and $\mathbf{r}$, and the modulus $\upsilon(t)$ of the relative velocity of the object and target. We differentiate the variables x, r and υ with respect to t (the derivative $\dot{x}$ being calculated using (1.6)). We put $w = (\mathbf{v}, \mathbf{W})/\upsilon$. We obtain the system of differential equations

$$\dot{x} = (\alpha r - \upsilon x)(1 - x^2)/(rx), \quad \dot{r} = -\upsilon x, \dot{\upsilon} = w \tag{2.1}$$

We consider w to be an unknown function of time which satisfies the constraint $|w(t)| \leqslant w_0$, where w_0 is a constant.

The vector equation (1.6) is equivalent to a sixth-order system of differential equations. It is convenient to study the behaviour of the trajectories of Eq. (1.6) as $t \to \infty$ using system (2.1). Suppose $\mathbf{r}(t)$, $\mathbf{v}(t)$ is some trajectory of Eq. (1.6), $t \geqslant t_0$, and $(\mathbf{r}(t_0), \mathbf{v}(t_0)) > 0$ at the initial time t_0. We consider the corresponding trajectory $(x(t), r(t), \upsilon(t))$ of system (2.1). Because $d = r(1-x^2)^{1/2} > 0$, we have $r(t) > 0$ and $x(t) < 1$ when $t \in [t_0, \infty)$.

The inequality $x(t) > 0$ is preserved when $t > t_0$, i.e. the projection of the relative velocity $\mathbf{v}$ along the radius vector $\mathbf{r}$ is always positive.

Proof. Assume the contrary. Let $t_1 > t_0$ be the earliest time at which $x(t_1) = 0$. Since $x(t)$, $r(t)$ is continuous with respect to t, $r(t_1) > 0$, and the velocity $\upsilon(t)$ is assumed to be bounded, t_2, $t_0 < t_2 < t_1$ exists such that $\alpha r(t) - \upsilon(t)x(t) > 0$ when $t_2 \leqslant t_1$. Then the derivative $\dot{x}(t)$ and the quantity $x(t)$ itself are positive when $t_2 \leqslant t < t_1$, which contradicts $x(t_1) = 0$. Hence $x(t) > 0$ when $t \geqslant t_0$

From this and from the second equation in (2.1) it follows that $\dot{r}(t) < 0$ along the trajectory, i.e. the distance between object and target decreases monotonically. It turns out that $r(t) \to 0$ as $t \to \infty$.

Proof. Assume the contrary. Let $r(t) \geqslant r_1 > 0$. By assumption $\upsilon(t) \leqslant \upsilon_1$. Hence an x_1, $0 < x_1 < x(t_0)$ exists such that $\alpha r_1 - \upsilon x > 0$ if $x \leqslant x_1$. Suppose that $t_1 > t_0$ is the earliest time at which $x(t) = x_1$, i.e. $x(t) > x(t_1) = x_1$, $t_0 \leqslant t < t_1$. From the choice of x_1 the right-hand side of the first equation in (2.1) at time t_1 is positive, so that the derivative $\dot{x}(t_1)$ is also positive, which contradicts the inequality $x(t) > x(t_1)$, $t < t_1$. Hence inequality $x(t) > x_1$ is preserved for all $t \geqslant t_0$. Then from the second equation in (2.1) we obtain $\dot{r}(t) = -\upsilon(t)x(t) \leqslant -\upsilon_0 x_1 < 0$, i.e. the function $r(t)$ decreases without limit. This contradiction proves that $r(t) \to 0$ as $t \to \infty$.

We will estimate the rate of approach of the object and target by differentiating the quantity $\psi(t) = \alpha r(t) - \upsilon(t)x(t)$ with respect to time. We obtain

$$d\psi/dt = -x(\alpha\upsilon + w) - \upsilon(1 - x^2)\psi/(rx)$$

We will assume that $\alpha\upsilon_0 > w_0$. This inequality can always be satisfied by choosing an appropriate α. Then $\dot{\psi}(t) < 0$ if $\psi(t) = 0$. From this it follows that the quantity $\psi(t)$ vanishes along the trajectory no more than once, when it changes sign from plus to minus. The sign of the derivative $\dot{x}$ is the same as the sign of ψ, and so the phase coordinate $x(t)$ increases when $\psi(t) > 0$ and decreases when $\psi(t) < 0$.

Let t_1 be such that $\psi(t) \leqslant 0$. It follows from the arguments given above that $\psi(t) < 0$ when $t > t_1$. Then $\alpha r(t) < \upsilon(t)x(t)$, and from the second equation of system (2.1) we have

$$\dot{r}(t) = -\upsilon(t)x(t) < -\alpha r(t), \quad t > t_1$$

From this we obtain an estimate for the time-dependence of the distance r

$$r(t) \leqslant r(t_1)e^{-\alpha(t-t_1)}, \quad t > t_1 \tag{2.2}$$

where t_1 is such that $\alpha r(t_1) \leqslant \upsilon(t_1)x(t_1)$.

3. A SPECIAL CASE

We consider the case $\mathbf{W} = 0$, i.e. the algorithm is used to guide the object when there are no external forces onto a target moving with a constant velocity in a straight line. Without loss of generality we will assume the target to be stationary (which can be achieved by a suitable change of variables). In this case Eq. (1.6) is equivalent to the system

$$\dot{\mathbf{v}} = -\alpha\mathbf{v} + \alpha\upsilon^2\mathbf{r}/(\mathbf{r}, \mathbf{v}), \quad \dot{\mathbf{r}} = -\mathbf{v} \tag{3.1}$$

Since $d\upsilon^2/dt = 2(\mathbf{v}, \dot{\mathbf{v}}) = 2(\mathbf{v}, \mathbf{u}) = 0$ we have $\upsilon = \text{const}$, and system (2.1) acquires the form

$$\dot{x} = (1 - x^2)(\alpha r - \upsilon x)/(r, x), \quad \dot{r} = -\upsilon x \tag{3.2}$$

where α, υ are constant coefficients.

We consider part of the phase plane of system (3.1): $r > 0$, $0 < x \leqslant 1$ (Fig. 2). Its vector field is constructed as follows: on the line $x = 1$ we have $\dot{x} = 0$, $r = -\upsilon$ and on the line $\alpha r = \upsilon x$ we have $\dot{x} = 0$, above this line $\dot{x} > 0$ and below $-\dot{x} < 0$. The system trajectory behaves as follows: the coordinate $r(t)$ strictly decreases and $r(t) \to 0$ as $t \to \infty$, the coordinate $x(t)$ increases up to the intersection with the line $\alpha r = \upsilon x$, and then decreases. It is easy to see that $x(t) \to 0$ when $t \to \infty$, as otherwise it follows from the second equation of (3.2) that $rt \to -\infty$.

It is clear from the structure of the vector field that for each trajectory starting above the line $\alpha r = \upsilon x$, there is a unique point of intersection with that line. Suppose $x_1 = x(t_1)$, $r_1 = r(t_1)$ are such that $\alpha r_1 \leqslant \upsilon_1 x_1$, i.e. the point (x_1, r_1) either lies on the line $\alpha r = \upsilon x$ or below it. Then the quantity $r(t)$ satisfies inequality (2.2) when $t \geqslant t_1$.

Let $x_0 = x(t_0)$, $r_0 = r(t_0)$ be the initial point of a trajectory situated above the line $\alpha r = \upsilon x$, and let t_1 be the time of intersection of the trajectory with this line. It is interesting to investigate how the time $t_1 - t_0$ taken to reach this line depends on x_0 and r_0.

We will derive two estimates.

The coordinate $x(t)$ increases along the trajectory above the line $\alpha r = \upsilon x$, and we therefore have $|\dot{r}(t)| = \upsilon x(t) \geqslant \upsilon x_0$, $t_0 \leqslant t \leqslant t_1$ (Fig. 2). Furthermore, $r_1 > \upsilon x_0/\alpha$. From this we obtain an estimate for the time taken to reach the line $\alpha r = \upsilon x$ from the point (x_0, r_0)

$$t_1 - t_0 \leq (r_0 - \upsilon x_0/\alpha)/(\upsilon x_0) = (\alpha r_0 - \upsilon x_0)/\alpha\upsilon x_0 \tag{3.3}$$

When $x_0 = 1$ this estimate is exact, but it becomes coarser as x_0 decreases.

We will obtain another estimate which is better for small x_0. We put $\xi(t) = \alpha r(t) - \upsilon x(t) + \upsilon x_0^2$. Because the section of the trajectory under consideration lies above the line $\alpha r = \upsilon x$, we have $\xi(t) > 0$, $t_0 \leqslant t \leqslant t_1$. We differentiate $\xi(t)$ with respect to time. Since $x_0 \leqslant x(t) \leqslant 1$, we have

$$\dot{\xi}(t) = -\upsilon(\alpha r - \upsilon x + \upsilon x^3)/(rx) \leqslant -\upsilon(\alpha r - \upsilon x + \upsilon x_0^2)/r_0 = -\upsilon\xi(t)/r_0 < 0$$

from which it follows that

$$t_1 - t_0 \leqslant r_0 \ln(\xi(t_0)/\xi(t_1))/\upsilon$$

Using $\xi(t_1) = \upsilon x_0^2$ we obtain yet another estimate for the time taken to reach the line $\alpha r = \upsilon x$

$$t_1 - t_0 \leqslant r_0 \ln((\alpha r_0 - \upsilon x_0)/(\upsilon x_0^2) + 1)/\upsilon \tag{3.4}$$

Figure 3 shows graphs of the dependence on x_0 of the right-hand sides of the estimates derived for the following parameter values: $\alpha = 0.5$, $r_0 = 100$, $\upsilon = 10$. The continuous and dashed lines correspond to estimates (3.3) and (3.4), respectively.

We will conclude the investigation of this special case by calculating the absolute magnitude of the control. From the first equation in (3.1) we obtain

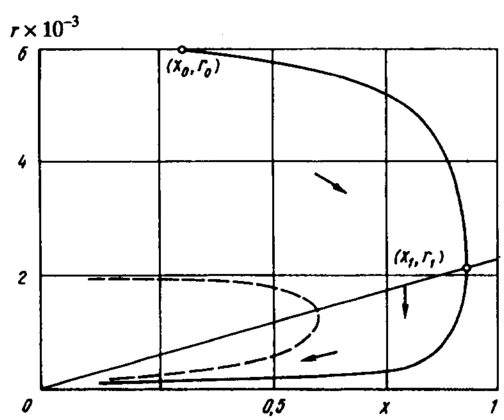

Fig. 2.

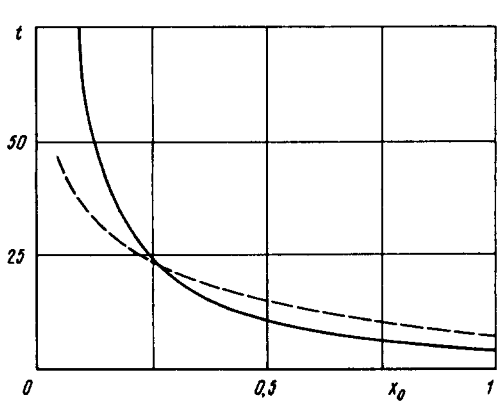

Fig. 3.

$$u^2 = (\dot{\mathbf{v}}, \dot{\mathbf{v}}) = \alpha^2 \upsilon^2 (1 - x^2)/x^2 = \alpha^2 \upsilon^2 \mathrm{tg}^2 \varphi \tag{3.5}$$

The quantity $u(t)$ therefore depends monotonically on x (and consequently on the angle φ): it decreases whenever x increases, i.e. up to the intersection of the trajectory with the line $\alpha r = \upsilon x$, and it then increases. If the condition $| u(t) | \leqslant u_0$ is imposed on the control, then $u(t)$ reaches the constraint when

$$x(t) = \alpha \upsilon (\alpha^2 \upsilon^2 + u_0^2)^{-1/2}$$

is satisfied, at which point it is impossible to continue the given algorithm with the previously given coefficient α.

4. OPTIMAL TARGET BEHAVIOUR

We return to the case $\mathbf{W} \neq 0$. The controlling acceleration of the object is governed by the guiding algorithm, whereas the behaviour of the target is arbitrary. How should one move the target, i.e. what should the acceleration $\mathbf{W}$ be, in order to "worsen" the approach? An answer to this question may be obtained by solving an optimal control problem (optimal from the point of view of the target). Various formulations of this problem are possible. We will consider one of them.

Suppose that the behaviour of the system is described by Eqs (2.1), where the control $w(t)$ is a piecewise-continuous function and is governed by the constraint

$$|w(t)| \leqslant w_0 \tag{4.1}$$

It is required to find a control law $w(t)$, $t_0 \leqslant t \leqslant T$ satisfying constraint (4.1) such that the distance $r(T)$ at the final time T of the process is maximized.

To solve this problem we use the Pontryagin maximum principle [3]. The Hamiltonian for system (2.1) has the form

$$H = p_1 (1 - x^2)(\alpha r - \upsilon x)/(rx) - p_2 \upsilon x + p_3 w$$

Here p_1, p_2, p_3 are the canonically conjugate momenta corresponding to the phase variables x, r and υ, respectively. We write out the conjugate equations

$$\dot{p}_1 = -p_1 (2\upsilon x^3 - \alpha r x^2 - \alpha r)/(rx^2) + p_2 \upsilon \tag{4.2}$$

$$\dot{p}_2 = -p_1 (1 - x^2)\upsilon / r^2, \quad \dot{p}_3 = p_1 (1 - x^2)/r + p_2 x$$

In this case the transversality condition takes the form

$$p_1(T) = p_3(T) = 0, \quad p_2(T) = 1 \tag{4.3}$$

From the maximum principle we obtain the following necessary condition for the optimality of the control $w(t)$

$$w(t) = w_0 \operatorname{sign} p_3 \tag{4.4}$$

It can be shown that

$$p_3(t) < 0, \quad t_0 \leqslant t < T \tag{4.5}$$

Then $w(t) = -w_0$.

To prove inequality (4.5) we differentiate $\dot{p}_3$ with respect to time and use (2.1) and (4.2). After some reduction we obtain a linear differential equation in $\dot{p}_3$

$$d\dot{p}_3/dt = \alpha (1 - x^2) x^{-2} \dot{p}_3 \tag{4.6}$$

It was shown in Section 2 that $x(t) > 0$. Hence the function $\alpha(1 - x^2)x^{-2}$ is bounded in the interval $[t_0, T]$ and Eq. (4.6) satisfies the solution uniqueness condition. Since this equation has a zero solution, all its other solutions have constant sign. From (4.3) and the final equation in (4.2) it follows that $\dot{p}_3(T) = x(T) > 0$. Then $\dot{p}_3(t) > 0$, and using $p_3(T) = 0$ we obtain (4.5).

We return to Eq. (1.6) describing the motion of the object relative to the target. In this equation the vector function $\mathbf{W}(t)$ can be considered as the control for the target. If the constraint $|\mathbf{W}(t)| \leqslant w_0$ is imposed on the function $\mathbf{W}(t)$, inequality (4.1) will follow from the definition of $w(t)$. This restriction is satisfied by the unique vector function $\mathbf{W}(t) = -w_0 \upsilon^{-1}(t)\mathbf{v}(t)$ which realizes the optimal control law $w(t) = -w_0$.

We will verify that the control obtained is optimal. Because the necessary condition for optimality—the maximum principle—is satisfied by a unique control, it is sufficient to show the existence of the solution of the given optimal control problem.

We will extend the class of admissible controls. We shall assume that $w(t)$ is a measurable function which satisfies constraint (4.1) almost everywhere on $[t_0, T]$, i.e. the set of admissible controls D is a sphere of radius w_0 in the space $L_\infty([t_0, T])$.

Suppose as before that the initial state of system (2.1) satisfies the inequalities $r(t_0) > 0$, $0 < x(t_0) < 1$, and that instead of the assumption $0 < \upsilon_0 \leqslant \upsilon(t) \leqslant \upsilon_1$, $t_0 \leqslant t \leqslant T$ we impose the constraint $\upsilon(t)_0 > w_0(T - t_0)$. From the arguments given in Section 2 it follows that for any $w(t) \in D$ a solution of the system of differential equations (2.1) exists in the interval $[t_0, T]$.

The variable $r(t)$ decreases along the solution of system (2.1), and so the functional $I(w) = -r(T)$ is bounded from below: $I(w) = -r(t_0)$. A minimizing sequence of admissible controls $w_i(t) \in D : I(w_i) > I(w_{i+1})$, $I(w_i) \to \inf_{w\in D} I(w)$ exists as $i \to \infty$. We know [4] that the sphere D is compact in the *-weak topology of the space $L_\infty([t_0, T])$, which is dual to the space $L_1([t_0, T])$. Hence a sequence w_{i_k} and a function $w_* \in D$ exist such that

$$\lim_{k\to\infty} \int_{t_0}^{T} g(s) w_{i_k}(s)ds = \int_{t_0}^{T} g(s) w_*(s) ds \forall g \in L_1([t_0, T]) \tag{4.7}$$

We denote by $y_k(t) = (x_k(t), r_k(t), \upsilon_k(t))$ the solution of system (2.1) corresponding to the control $w_{i_k}(t)$. Since $0 < r_k(t) < r(t_0)$, $0 < x_k(t) < 1$, $\upsilon(t_0) - w_0(T - t_0) \leqslant \upsilon_k(t) \leqslant \upsilon(t_0) + w_0(T - t_0)$ for all k, the sequence of functions $y_k(t)$ is uniformly bounded on $[t_0, T]$. It can be shown that it is also equicontinuous.

It is sufficient to prove the uniform boundedness of the right-hand side of the system on the trajectories $y_k(t)$. The variables $r(t)$ and $x(t)$ are connected by the relation $r(t)(1 - x^2(t))^{1/2} = d(t)$, and the quantity d satisfies Eq. (1.2). Hence, for any k

$$r_k(T)(1 - x_k^2(T))^{1/2} = r(t_0)(1 - x^2(t_0))^{1/2} \exp(-\alpha(T - t_0)) \tag{4.8}$$

Note that the right-hand side of Eq. (4.8) is a constant quantity, depending only on the initial position of the system. The sequence $I(w_{i_k}) = -r_k(T)$ decreases as k increases, and the function $r_k(t)$ decreases with t, and so

$$0 < r_1(T) \leqslant r_k(T) \leqslant r_k(t), \quad k = 1,2,\ldots, \quad t_0 \leqslant t \leqslant T \tag{4.9}$$

From this and from (4.8) we obtain

$$x_1(T) \leqslant x_k(T) \tag{4.10}$$

It was shown in Section 2 that the coordinate $x(t)$ increased when $\psi(i) > 0$ and decreased when $\psi(i) < 0$, and that $\psi(t)$ vanished no more than once, when it changed sign from plus to minus. Hence, for any trajectory of system (2.1) the inequality $\min(x(t_0), x(T)) \leqslant x(t)$, $t_0 \leqslant t \leqslant T$ is satisfied. Using (4.10) we obtain $0 < \min(x(t_0), x_1(T)) \leqslant x_k(t)$, $k = 1, 2, \ldots, t_0 \leqslant t \leqslant T$. Thus the denominator $x_k(t)r_k(t)$ on the right-hand side of the equation for the derivative of $\dot{x}$ is governed by the constraint $0 < r_1(T) \min(x(t_0), x_1(T)) \leqslant x_k(t)r_k(t)$, $k = 1, 2, \ldots, t_0 \leqslant t \leqslant T$. Consequently, the right-hand side of system (2.1) is uniformly bounded in k, t along the trajectories $y_k(t)$, and the sequence of functions $y_k(t)$ is equicontinuous.

By Artsel's theorem a subsequence $y_{k_j}(t)$ exists that converges uniformly to some function $y_*(t)$. We verify that the function $y_*(t)$ is a solution of system (2.1) with control $w_*(t)$. We represent system (2.1) with control $w_k(t)$ in the form

$$y_k(t) = y(t_0) + \int_{t_0}^{t} f(y_k(\tau))d\tau + \int_{t_0}^{t} w_k(\tau)d\tau$$

where the vector function $f(y)$ denotes the right-hand side of the system without the control. It is clear that

$$\lim_{k\to\infty}\int_{t_0}^{t} f(y_k(\tau))d\tau=\int_{t_0}^{t} f(y_*(\tau))d\tau$$

The equality

$$\lim_{k\to\infty}\int_{t_0}^{t} w_k(\tau)d\tau=\int_{t_0}^{t} w_*(\tau)d\tau$$

follows from Eq. (4.7) in which one must put

$$g(\tau)=\begin{cases}1, & \text{if } t_0\leqslant\tau\leqslant t\\ 0, & \text{if } t<\tau\leqslant T\end{cases}$$

The function $y_*(t)$ is therefore a solution of system (2.1) with control $w_*(t)$. Because $\{w_i\}$ is a minimizing sequence, y_*, w_* is an optimal process, i.e. $I(w_*) = \inf_{w\in D} I(w)$.

There is thus a solution on the set D of the optimal control problem under consideration. The optimal control is unique because it is uniquely defined by the maximum principle—a necessary condition for optimality—and has the form $w_*(t) = -w_0$ almost everywhere on $[t_0, T]$. It is clear that the function $w_*(t) = -w_0$ is also the unique optimal control in the class of piecewise-continuous functions.

5. NUMERICAL RESULTS

Figures 2, 4 and 5 show the results of calculations for the special case $\mathbf{W} = 0$. Equation (1.6) and system (3.2) were integrated numerically using the Runge–Kutta method with the following parameter values: $\alpha = 0.5$,

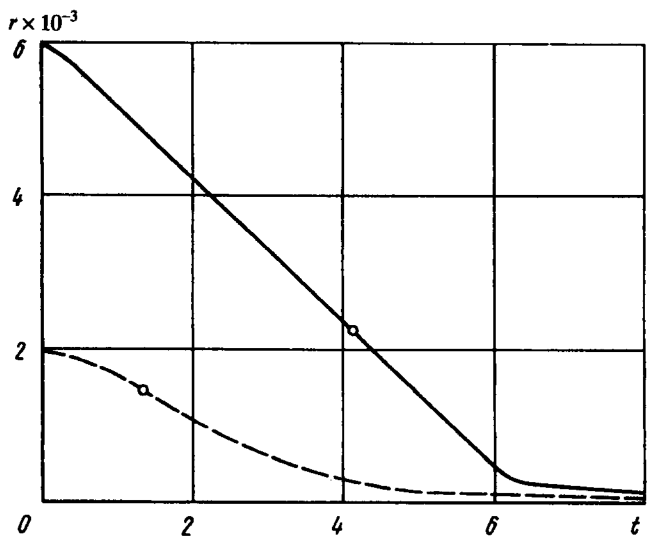

Fig. 4.

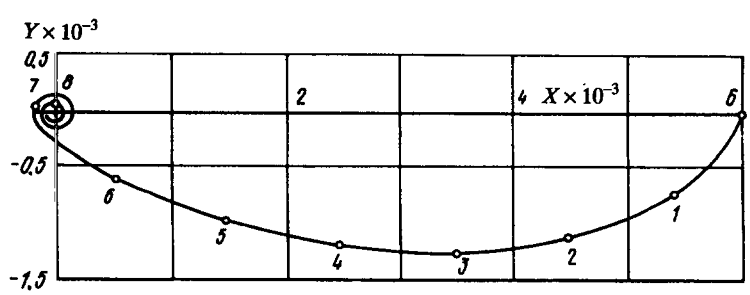

Fig. 5.

$\upsilon = 1000$. The continuous lines correspond to trajectories leaving the point $\mathbf{r} = (6000, 0, 0)$, $\dot{\mathbf{r}} = (-300, -954, 0)$ in the phase space $\mathbf{r}$, $\dot{\mathbf{r}}$, while the dashed lines are trajectories leaving the point $\mathbf{r} = (2000, 0, 0)$, $\dot{\mathbf{r}} = (-100, -995, 0)$. Here $\mathbf{r}$ is a radius vector giving the position of the object in an inertial system of coordinates with the target at the origin. In phase variables x, r the corresponding initial conditions for system (3.2) have the form $r = 6000, x = 0.3$ and $r = 2000, x = 0.1$.

Figure 2 shows trajectories of system (3.2) in the phase space x, r, while Fig. 4 is a graph of the time-dependence of the distance between the object and target; the points denote instants of intersection of the trajectory with the line $\alpha r = \upsilon x$. Figure 5 shows the initial segment of the trajectory of motion of the object in the OXY plane of the inertial system of coordinates; the points show the position of the object at the indicated times.

We wish to thank A. I. Ovseyevich for useful discussions.

This research was performed with the financial support of the Russian Foundation for Basic Research (93-013-16286) and the International Science Foundation (M4F000).

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Translated by R.L.Z.